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Bianchi Type III Foliation of the de Sitter Space

P. Crawford[1] and P. Vargas Moniz[1]

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We show that the de Sitter space can be foliated into spacelike hypersurfaces with locally rotationally symmetric Bianchi type III properties. This proves as well that a particular Bianchi type III solution is isotropic. In addition, we apply a criterion of isotropy, based on the vanishing of the Weyl tensor, which is particularly suited for cosmological models where there is not a preferred timelike vector field.

The de Sitter space is a maximally symmetric spacetime, where there is not a preferred timelike direction (Weinberg, 1972; Ryan and Shepley, 1975). This fact should allow us to define different types of spacelike foliations. It is well known that the de Sitter space can be represented in all the three standard forms of the Robertson–Walker spacetimes. In addition, Torrence and Couch (1988) have shown that the de Sitter space also admits foliations of the Kantowski–Sachs type. Therefore, it will be important to find out whether the de Sitter space admits other foliations, with different properties from the ones mentioned above.

We present a particular locally rotationally symmetric (LRS) Bianchi type III model (Collins, 1977; MacCallum, 1979*a*, *b*) with a cosmological constant. It will be shown below that this Bianchi type III solution is a portion of the de Sitter space and hence it is isotropic. However, this isotropic solution has a nonzero shear tensor. Such a conflicting situation also arises between some results due to Grøn (1986) and to Torrence and Couch (1988) on an empty Kantowski–Sachs universe with a cosmological constant. The right interpretation lies in the fact that for specific cases the shear tensor of the normal congruence to the homogeneous hypersurfaces cannot provide reliable information about the degree of isotropy. Such situations occur when a preferred timelike direction field does not exist. Consequently, we

[1]Departamento de Física, Universidade de Lisboa, Campo Grande, Ed-C1, P-4, 1700-Lisboa, Portugal.

have proposed (Crawford and Vargas Moniz, 1992*a*, *b*) a frame-independent requirement to evaluate the isotropy based on the vanishing of the Weyl tensor. This criterion is discussed here. We calculate as well the Killing vectors which form the Lie algebra of the isometry group of this LRS Bianchi type III solution and find that there are ten. Only four of the ten Killing vectors generate symmetries on the hypersurfaces $\{t = \text{const}\}$. Thus, this case corresponds rather to a Bianchi type III foliation of the de Sitter space, as will be shown.

As a starting point for our discussion, let us write the family of LRS Bianchi type III metrics as

$$ds^2 = -dt^2 + A^2(t)\, dr^2 + B^2(t)[d\theta^2 + \sinh^2\theta\, d\varphi^2] \tag{1}$$

We are interested in metrics (1) which satisfy the Einstein equations

$$R_{ab} = \Lambda g_{ab}; \qquad \Lambda > 0 \tag{2}$$

Note that any solution which satisfies equation (2) is also designated as an Einstein space, where the de Sitter space is only a particular case (Rindler, 1979). The Einstein equations (2) for the metric (1) are

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2 - 1}{B^2} = \Lambda \tag{3}$$

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2 - 1}{B^2} = \Lambda \tag{4}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = \Lambda \tag{5}$$

Integration of equation (4) gives

$$\dot{B}^2 = \left(\frac{\Lambda}{3} B^2 - 1 + \frac{C}{B}\right) \tag{6}$$

where C is a constant of integration. Putting $C = 0$, integrating once more, and since (3) and (4) imply $A(t) \propto \dot{B}$, it is straightforward to obtain

$$A(t) = \frac{1}{H_0}\cosh(H_0 t), \qquad B(t) = \frac{1}{H_0}\sinh(H_0 t) \tag{7}$$

where $H_0 = (\Lambda/3)^{1/2}$. This case describes a universe that emerges from a cigar singularity: $B \to 0$, $A \to A_0 > 0$ as $t \to 0$. The general solutions of equations (1), (2) can be found in Stewart and Ellis (1968), Cahen and Defrise (1968), Mossiaux *et al.* (1981), MacCallum *et al.* (1982), and Lorenz (1982, 1983). The particular solution (1), (7) was communicated by Baofa (1991), but unfortunately it was announced as a Kantowski–Sachs universe model. The

Kantowski-Sachs case was previously reported by Grøn (1986). The LRS Bianchi type III universe (1), (7) has a volume expansion given by

$$\Theta = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} = H_0[\tanh(H_0 t) + 2\coth(H_0 t)] \tag{8}$$

and the shear invariant σ ($\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij}$) is

$$\sigma = \frac{1}{\sqrt{3}}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = -\frac{2H_0}{\sqrt{3}\sinh(2H_0 t)} \tag{9}$$

The volume expansion (8) tends toward $3H_0$ and the shear decays exponentially toward zero when $H_0 t \gg 1$.

The solution (1), (7) has, however, a six-parameter isotropy group defined at every point and hence it is isotropic. This is shown as follows. The de Sitter space S_4^- (Rindler, 1979) is defined through the hypersurface

$$X^2 + Y^2 + Z^2 + W^2 - \tilde{T}^2 = \alpha^2 \tag{10}$$

embedded in the 5-dimensional Minkowski spacetime M_5,

$$ds^2 = -d\tilde{T}^2 + dX^2 + dY^2 + dZ^2 + dW^2 \tag{11}$$

where $\alpha = H_0^{-1}$. Using the coordinates defined by

$$\begin{aligned} X &= \alpha \sinh\theta \cos\phi \tan T \\ Y &= \alpha \sinh\theta \sin\phi \tan T \\ Z &= \alpha \cos R \sec T \\ W &= \alpha \sin R \sec T \\ \tilde{T} &= \alpha \cosh\theta \tan T \end{aligned} \tag{12}$$

we find that the de Sitter metric induced by this embedding becomes

$$ds^2 = \frac{1}{\sin T}[-dT^2 + dR^2 + \cos T\,(d\theta^2 + \sinh\,\theta\, d\phi^2)] \tag{13}$$

The introduction of a new cosmic time t defined by

$$\frac{1 + \sin T}{\cos T} = e^{t/\alpha} \tag{14}$$

gives the metric

$$ds^2 = -dt^2 + \alpha^2 \sinh^2(t/\alpha)\, dR^2 + \alpha^2 \cosh^2(t/\alpha)\, [d\theta^2 + \sinh^2\theta\, d\phi^2] \tag{15}$$

This is the metric form (1) with the scale factors given by (7). Thus, the solution (7) is isotropic as it is mapped into the de Sitter space.

In order to classify the universe described by equations (1), (7) according to its intrinsic geometrical properties, we compute its Killing vectors. Following the guidelines in Crawford and Vargas Moniz (1992*b*), we find ten Killing vectors, namely,

$$K_1 = \cos\phi \frac{\partial}{\partial\theta} - \coth\theta \sin\phi \frac{\partial}{\partial\phi} \tag{16}$$

$$K_2 = \sin\phi \frac{\partial}{\partial\theta} + \coth\theta \cos\phi \frac{\partial}{\partial\phi} \tag{17}$$

$$K_3 = \alpha \cosh\theta \cos r \frac{\partial}{\partial t} - \cosh\theta \sin r \tanh\left(\frac{t}{\alpha}\right) \frac{\partial}{\partial r} - \sinh\theta \cos r \coth\left(\frac{t}{\alpha}\right) \frac{\partial}{\partial\theta} \tag{18}$$

$$K_4 = \alpha \sin r \cosh\theta \frac{\partial}{\partial t} + \cosh\theta \cos r \tanh\left(\frac{t}{\alpha}\right) \frac{\partial}{\partial r} - \sinh\theta \sin r \coth\left(\frac{t}{\alpha}\right) \frac{\partial}{\partial\theta} \tag{19}$$

$$K_5 = \frac{\partial}{\partial\phi} \tag{20}$$

$$K_6 = -\alpha \sinh\theta \sin\phi \cosh r \frac{\partial}{\partial t} + \sinh\theta \sin\phi \sin r \tanh\left(\frac{t}{\alpha}\right) \frac{\partial}{\partial r} + \cosh\theta \sin\phi \cos r \tanh\left(\frac{t}{\alpha}\right) \frac{\partial}{\partial\theta} + \frac{1}{\sinh\theta} \cos\phi \cos r \coth\left(\frac{t}{\alpha}\right) \frac{\partial}{\partial\phi} \tag{21}$$

$$K_7 = \cos\phi \frac{\partial}{\partial\theta} - \cot\theta \sin\phi \frac{\partial}{\partial\phi} \tag{22}$$

$$K_8 = -\alpha \sinh\theta \cos\phi \sin r \frac{\partial}{\partial t} - \sinh\theta \cos\phi \cos r \tanh\left(\frac{t}{\alpha}\right) \frac{\partial}{\partial r} + \cosh\theta \cos\phi \sin r \coth\left(\frac{t}{\alpha}\right) \frac{\partial}{\partial\theta} - \frac{1}{\sinh\theta} \sin\phi \sin r \coth\left(\frac{t}{\alpha}\right) \frac{\partial}{\partial\phi} \tag{23}$$

$$K_9 = -\alpha \sinh\theta \sin\phi \sin r \frac{\partial}{\partial t} - \sinh\theta \sin\phi \cos r \tanh\left(\frac{t}{\alpha}\right)\frac{\partial}{\partial r}$$

$$+\cosh\theta \sin\phi \sin r \coth t \frac{\partial}{\partial\theta} + \frac{1}{\sinh\theta}\cos\phi \sin r \coth\left(\frac{t}{\alpha}\right)\frac{\partial}{\partial\phi} \tag{24}$$

$$K_{10} = \frac{\partial}{\partial r} \tag{25}$$

It is not difficult to verify that all these vector fields are linearly independent and are all solutions of Killing's equation

$$\mathscr{L}_{K_{(A)}}\mathbf{g} = 0$$

for the metric (15), where $\mathscr{L}$ stands for the Lie derivative. As expected, the space represented by this metric admits a maximal group G_{10} of isometries. The symmetries of the 3-surfaces $\{t=\text{const}\}$ are generated only by the subalgebra formed by the four Killing vectors $\{K_1, K_2, K_5, K_{10}\}$. More exactly, $\{K_1, K_2, K_5\}$ generate the subalgebra of a $G_3 VIII$ that acts multitransitively on the two-dimensional orbits $d\theta^2 + \sinh^2\theta\, d\varphi^2$ on which $\{t, r=\text{const}\}$. The Killing vector field K_{10} is the generator of the spatial translations and it must be nonnull everywhere so that the orbits of G_4 be three-dimensional. Also, from the Jacobi identities it follows that $[K_{10}, K_i]=0$, where $i=1, 2, 5$. The Bianchi III type Lie subalgebra, of the G_3 subgroup that acts transitively on the homogeneous hypersurfaces $\{t=\text{const}\}$, is made up by the Killing vector fields $\{K_1, K_2+K_5, K_{10}\}$. This proves our point that the metric (15) can only represent a foliation of de Sitter space of an LRS Bianchi III type.

The apparent conflict between a nonzero shear solution which is maximally symmetric, and therefore has a six-parameter isotropy group at every point, has been cleared up by the present authors (Crawford and Vargas Moniz, 1992*a*, *b*). In what follows we present a brief explanation. The shear tensor changes according to the different frames we may choose to depict the different foliations. Thus, when a preferred timelike direction cannot be selected, the shear criterion is dubious and might insinuate a misleading interpretation of isotropy. The applicability of the shear tensor should be restricted to spacetimes with a preferred timelike vector field, like the ones with a perfect fluid matter content. In the present case (an Einstein space) we can choose *different* but *completely equivalent* timelike directions, in the sense that all observers will measure the same constant Λ. In particular, there must exist at least one particular frame where the shear will be zero. Hence, the exponential decay of the shear when $H_0 t \gg 1$ does not represents an isotropization from a shear-dominated Bianchi type III universe toward the isotropic de Sitter space S_4^-: the solution (1), (7) is actually (a portion of) the de Sitter space, i.e., it is already isotropic.

As far as the shear is concerned, it gives valuable information about the specific foliation. The model with metric (1), (7) constitutes a de Sitter space where a particular timelike direction was introduced. Observers whose worldlines are the integral curves of the vector field $\partial/\partial t$ observe a universe (1), (7) with a nonzero shear tensor. In this sense, we could say that the foliation is spatially anisotropic, but not the solution. Thus, the appropriate classification of the solution (7) is a spatially anisotropic (nonzero shear) foliation of the expanding de Sitter universe in a rather bizarre coordinate frame (of the LRS Bianchi III type) which conceals its intrinsic geometric properties.

Complying with the standard definition of isotropy (Wald, 1984), we ought to have a frame-independent criterion. Thus, we proposed (Crawford and Vargas Moniz, 1992*a*, *b*) that the relevant requirement in terms of isotropy is the vanishing of the Weyl tensor, as it is independent of the particular spacetime foliation. This proves to be of particular relevance for studying the isotropy of spaces where a preferential timelike vector field does not exist. The Weyl tensor for the metrics (1), (2) vanishes if and only if (Crawford and Vargas Moniz, 1992*a*, *b*)

$$\frac{\dot{A}\dot{B}}{AB}-\frac{\dot{B}^2-1}{B^2}=0 \tag{26}$$

This equation is satisfied only for A and B given by (7). This also proves that the solution (7) is the unique isotropic LRS Bianchi type III spacetime. All the other solutions ($C\neq 0$) are unequivocally anisotropic. We also remark that the Riemann curvature components R^{ab}_{cd} for the metric (1), (7) are all identical, which indeed confirms that this solution represents a spacetime of constant curvature.

Summarizing, our purposes have been twofold. On the one hand, our calculations show that the de Sitter space can be foliated into spacelike hypersurfaces with LRS Bianchi type III properties. Moreover, we identified the unique LRS Bianchi type III solution that can be mapped in the de Sitter space, and hence it is isotropic. On the other hand, an alternative criterion of isotropy of spaces was applied here. This requirement is particularly relevant when a preferred timelike vector field cannot be defined at all. Instead of the quite common criterion of a vanishing shear tensor, or, more specifically, $\sigma/\Theta\rightarrow 0$, which depends on the choice of the timelike direction, we required the vanishing of the Weyl tensor.

Einstein spaces [cf. equation (2)] can be considered as "vacuum-dominated" universes, where a preferred timelike vector field cannot be defined. This type of situation is interesting from the point of view of cosmological models which develop toward an inflationary stage of a de Sitter type.

In the case where the matter content is insignificant, the shear tensor of the normal congruence to the homogeneous hypersurfaces cannot provide reliable information about the degree of isotropy. Consequently, we used a frame-independent requirement to evaluate the isotropy, that is, the vanishing of the Weyl tensor. An Einstein space with a null Weyl tensor is a maximally symmetric spacetime, and thus it is isotropic.

Finally, it is interesting to note that the known de Sitter space foliations (including the one herewith described) are characterized by spacelike hypersurfaces of constant curvature. The foliations of the Robertson–Walker type correspond to 3-dimensional hypersurfaces of constant positive, negative, and zero curvature and the foliations of the Kantowski–Sachs and LRS Bianchi type III correspond to 2-dimensional surfaces of positive and negative curvature, respectively. Furthermore, applying the null Weyl criterion to the LRS Bianchi type I model, one gets the flat Robertson–Walker case. Hence, it will be interesting to study if the de Sitter space only admits foliations into spacelike hypersurfaces of constant curvature. If that is the case, what are the several possibilities to accomplish such a situation? This query will be analyzed in a future report.

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